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A Characterization of First-Order Phase Transitions for Superstable Interactions in Classical Statistical Mechanics

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We give bounds on finite-volume expectations for a set of boundary conditions containing the support of any tempered Gibbs state and prove a theorem connecting the behavior of Gibbs states to the differentiability of the pressure for continuum statistical mechanical systems with long-range superstable potentials. Convergence of grand canonical Gibbs states is also studied.

KEY WORDS: Differentiability of the pressure; Gibbs states; first-order phase transition; superstable interaction; continuum model.

1. INTRODUCTION

For a grand canonical system of particles, a first-order phase transition is said to occur if the pressure is not continuously differentiable with respect to chemical potential. First-order phase transitions are also generally associated with multiple infinite-volume Gibbs states. The existence of multiple Gibbs states, however, does not imply a first-order phase transition, as can be seen in the case of the two-dimensional Ising antiferromagnet (or, more appropriately, the equivalent lattice gas.[8] Rigorous connections, for lattice models, between the behavior of Gibbs states and the differentiability of the pressure or free energy with respect to various parameters have been made by a number of authors.[1-7]

In this paper we consider long-range, superstable interactions in $\mathbf{R}^d$. We prove that a first-order phase transition occurs at a point in phase space if and only if multiple, translation-invariant, tempered Gibbs states

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exist at that point and they yield strictly different expectations for the density of particles. An analogous statement is proven for differentiation with respect to the inverse temperature. Our results therefore extend to a broad class of continuum models a rigorous mathematical connection between two widely used criteria to establish phase transitions. To prove the main theorem, we show how finite-volume expectations of particle density and energy may be bounded in the presence of an arbitrary external configuration in the support set of any tempered Gibbs state. We also prove a convergence result for grand canonical, tempered Gibbs states when the respective temperatures or chemical potentials converge.

We note that the conclusions of our main theorem are known for a large class of lattice models with compact configuration space and bounded Hamiltonians.[6,7,3] The methods used in those references are not available here, since our Hamiltonians are unbounded and configurations of particles may have arbitrarily large local densities. Instead we use measure-theoretic techniques and especially the probability estimates of Ruelle.[9] Lebowitz and Presutti[2] obtained somewhat related results, using different methods, for models with unbounded spin spaces, but the conditions they impose on the Hamiltonian are not satisfied by the usual models of classical continuum statistical mechanics.

An extended version of this paper is available from the authors on request.

2. NOTATION AND PRELIMINARY RESULTS

For a Borel measurable subset $\Lambda \subset \mathbb{R}^d$ let $X(\Lambda)$ denote the set of all locally finite subsets of Λ. $X(\Lambda)$ represents configurations of identical particles in Λ. We let $\varnothing$ debote the empty configuration. Let B_Λ be the σ-field on $X(\Lambda)$ generated by all sets of the form $\{s \in X(\Lambda): |s \cap B| = m\}$, where B runs over all bounded Borel subsets of Λ, m runs over the set of nonnegative integers, and $|\cdot|$ denotes cardinality. We let $(\Omega, S) = (X(\mathbf{R}^d), B_{\mathbf{R}^d})$. For a configuration $x \in \Omega$ let $x_\Lambda = x \cap \Lambda$.

A Hamiltonian H is an S measurable map from the set of finite configurations Ω_F in Ω to $(-\infty, \infty]$ of the form

$$H(x) = \sum_{i<j} \varphi(x^i, x^j) - h|x| \tag{2.1}$$

where the function φ is a pair potential and where $h \in \mathbf{R}$. The configuration x in (2.1) is coordinatized by $x = \{x^1, x^2,..., x^{|x|}\}$. For $x \in X(\Lambda)$ we will sometimes write $H_\Lambda(x)$ instead of $H(x)$.

For a bounded Borel set Λ let $|\Lambda|$ denote the Lebesgue measure of Λ.

The symbol $|\cdot|$ may therefore represent cardinality or Lebesgue measure, but the meaning will always be clear from the context.

Define the interaction energy between $x \in X(\Lambda)$ and $s \cap \Lambda^c$ by

$$W_\Lambda(x \mid s) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(x^i, s^j) \tag{2.2}$$

where $x = \{x^1,..., x^n\}$, and $s \cap \Lambda^c = \{s^1,..., s^m\}$. We will sometimes write $W(x \mid s)$ when x and s are located in disjoint regions. Define

$$H_\Lambda(x \mid s) = H_\Lambda(x) + W_\Lambda(x \mid s) \tag{2.3}$$

For each $i \in \mathbf{Z}^d$ let

$$Q_i = \{r \in \mathbf{R}^d : r^k - 1/2 \leqslant i^k < r^k + 1/2, k = 1,..., d\}$$

so that the unit cubes $\{Q_i\}$ partition $\mathbf{R}^d$. Define $|x_i| \equiv |x_{Q_i}| = |x \cap Q_i|$. For a nonnegative integer k let Λ_k be the hypercube of length $2k-1$ centered at the origin in $\mathbf{R}^d$; Λ_k is then a union of $(2k-1)^d$ unit cubes of the form Q_i. We will also sometimes regard Λ_k as a subset of $\mathbf{Z}^d$ by letting Λ_k represent $\Lambda_k \cap \mathbf{Z}^d$.

For $i \in \mathbf{Z}^d$ or $\mathbf{R}^d$, let $\|i\| = \|(i^1,..., i^d)\| = \max_k |i^k|$ be the supnorm.

We assume throughout this paper that H satisfies the following conditions:

(a) H, φ are translation invariant.

(b) H is superstable,[9,10] i.e., there exist $A > 0$, $B \geqslant 0$ such that if the configuration x is contained in Λ_k for some k, then

$$H(x) \geqslant \sum_{i \in \Lambda_k} A |x_i|^2 - B |x_i| \tag{2.4}$$

(c) $H(x)$ is lower regular. There exists a positive function ψ on the nonnegative integers such that $\psi(m) \leqslant K m^{-\lambda}$ for $m \geqslant 1$, and for any Λ_1 and Λ_2 which are each finite unions of unit cubes of the form Q_i, with $x \subset \Lambda_1$ and $s \subset \Lambda_2$,

$$W(x \mid s) \geqslant -\sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \psi(\|i-j\|) |x_i| \cdot |s_j| \tag{2.5}$$

where $K > 0$, $\lambda > d$ are fixed.

(d) $H(x)$ is tempered. There exists $R_0 > 0$ such that, with the same notation as in part (c), assuming Λ_1 and Λ_2 are separated by a distance R_0 or more,

$$W(x \mid s) \leqslant K \sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \|i-j\|^{-\lambda} |x_i| \cdot |s_j| \tag{2.6}$$

We next define a measure for each bounded Borel set of $\mathbf{R}^d$. Let $X_N(\Lambda) \subset X(\Lambda)$ be the set of configurations of cardinality N in Λ and let $T: \Lambda^N \to X_N(\Lambda)$ be the map which takes the ordered N-tuple $(x_1,..., x_N)$ to the (unordered) set $\{x_1,..., x_N\}$. In a natural way T defines an equivalence relation on Λ^N and $X_N(\Lambda)$ may be regarded as the set of equivalence classes induced by T. For $N = 1, 2, 3,...$ let $d^N x$ be the projection of nd-dimensional Lebesgue measure onto $X_N(\Lambda)$ under the projection T: $\Lambda^N \to X_N(\Lambda)$. The measure $d^0 x$ assigns mass 1 to $X_0(\Lambda) = \{\varnothing\}$. Define $d^N x$ to be the zero measure on $X_M(\Lambda)$ for $M \neq N$. On $X(\Lambda) = \bigcup_{n=0}^{\infty} X_n(\Lambda)$

$$\nu_\Lambda(dx) = \sum_{n=0}^{\infty} \frac{d^n x}{n!}$$

If $\Lambda \cap A = \varnothing$, where Λ and A are Borel sets, then $(X(\Lambda), B_\Lambda, \nu_\Lambda) \times (X(A), B_A, \nu_A)$ may be identified with $(X(\Lambda \cup A)$, $B_{\Lambda \cup A}$, $\nu_{\Lambda \cup A})$ via $x_\Lambda \times x_A = x_\Lambda \cup x_A$. In particular, for any bounded Borel set Λ,

$$(\Omega, S) = (X(\Lambda), B_\Lambda) \times (X(\Lambda^c), B_{\Lambda^c}) \tag{2.7}$$

Let $\tilde{B}_\Lambda$ denote the inverse projection of B_Λ under the identification (2.7), so that $\tilde{B}_\Lambda$ is a σ-field on Ω.

Let Λ be a bounded Borel set in $\mathbf{R}^d$ and let s be a configuration in Λ^c. The finite-volume Gibbs state with boundary configuration s for H, $\beta > 0$, and h is

$$\mu_\Lambda(dx \mid s) = \frac{\exp\{-\beta H(x \mid s)\}}{Z_\Lambda(s)} \nu_n(dx) \tag{2.8}$$

where $Z_\Lambda(s) \equiv Z_\Lambda(\beta, h, s)$ makes $\mu_\Lambda(dx \mid s)$ a probability measure and β is inverse temperature. When $s = \varnothing$, let $\mu_\Lambda(dx \mid \varnothing) \equiv \mu_\Lambda(dx)$.

The pressure $p(\beta, h)$ for H is given by

$$\beta p(\beta, h) \equiv P(\beta, h) = \lim_{k \to \infty} \frac{\ln Z_{\Lambda_k}(\varnothing)}{|\Lambda_k|} \tag{2.9}$$

Remark 2.1. The limit in (2.9) is well known to exist[9,10] and to be a convex function of β and h for the models that we consider, and it is also possible to consider more general limits than described above, but this is as much as we will need.

Let $\{\pi_\Lambda\}$ denote the specification associated with β, h, and the Hamiltonian H (see ref. 3, p. 16) defined by

$$\pi_\Lambda(A \mid s) = \int_{A'} \mu_\Lambda(dx \mid s) \tag{2.10}$$

where $A' = \{x \in X(\Lambda)\colon x \vee s \in A\}$. This specification is defined with respect to the sets $\{R_\Lambda\}$ as defined by Preston and is consistent.[3]

A probability measure μ on Ω is a Gibbs state (or infinite-volume Gibbs state) for H, β, and h if

$$\mu(\pi_\Lambda(A \mid s)) = \mu(A)$$

for every $A \in S$ and every bounded Borel set Λ.

Folllowing Ruelle,[9] we define a Gibbs state μ to be tempered if μ is supported on

$$V_\infty = \bigcup_{N=1}^{\infty} V_N$$

where $V_N = \{x \in \Omega\colon \sum_{i \in \Lambda_k} |x_i|^2 \leqslant N^2 |\Lambda_k| \text{ for all } k\}$. The following proposition collects some results proved by Ruelle in ref. 9.

Proposition 2.1 (Ruelle[9]). Let Λ be a finite union of unit cubes of the form Q_i. Suppose $\tilde{\Lambda} \supset \Lambda$ is a bounded Borel set in $\mathbf{R}^d$. There exist constants $\gamma > 0$ and δ, depending only on β and h (independent of $\tilde{\Lambda}$ and Λ), such that the probability that $|x_\Lambda| \geqslant N|\Lambda|$ with respect to $\mu_{\tilde{\Lambda}}(dx \mid \varnothing)$ is less than $\exp[-(\gamma N^2 - \delta)/\Lambda|]$. The same probability estimate holds when $\mu_{\tilde{\Lambda}}(dx \mid \varnothing)$ is replaced by any tempered Gibbs state for β, h. Moreover, for any β, h, the set of translation-invariant, tempered Gibbs states is nonempty.

With Proposition 2.1 it is possible to describe another support set for tempered Gibbs states. Let $\ln_+ r = \max\{1, \ln r\}$. Define

$$\begin{aligned} U_n &= \{s \in \Omega\colon |s_i| \leqslant n(\ln_+ \|i\|)^{1/2} \text{ for all } i \in \mathbf{Z}^d\} \\ U_\infty &= \bigcup_{n=1}^{\infty} U_n \end{aligned} \tag{2.11}$$

A straightforward argument[2,11] shows that $\mu(U_\infty) = 1$ for any tempered Gibbs state μ.

The following lemma, stated without proof, will be used to control the effect of boundary configurations on certain expected values in the next section.

Lemma 2.1. Let $\varepsilon > 0$ and $s \in U_n$. Then for all k sufficiently large, the following hold:

(a) $W_{\Lambda_k}(x \mid s) \geqslant -D_k(s)|x_{\partial\Lambda_k}| - \varepsilon n |x_{\Lambda_m}|$.

(b) $|W_{\Lambda_k}(x \cap \Lambda_m \mid s)| \leqslant \varepsilon n |x_{\Lambda_m}|$.

Here m is the greatest integer $\leqslant k - C_\varepsilon(\ln k)^{1/(\lambda-d)}$, C_ε is a constant for each ε independent of k, $\partial\Lambda_k = \Lambda_k \backslash \Lambda_m$, and $D_k(s) \leqslant Cn(\ln k)^{1/2}$ for some constant C.

Rermark 2.2. It is also true that $W_{\Lambda_k}(x\,|\,s) \geqslant -D_k(s)\,|x_{\Lambda_k}|$ for all k.

For the convenience of the reader we conclude this section with two known results from measure theory which we will use in the next section. The first is a generalization of the Lebesgue dominated convergence theorem.[12]

Proposition 2.2. Let (X, B) be a measurable space and $\{\mu_n\}$ a sequence of measures on B that converge setwise to a measure μ. Let $\{f_n\}$ be a sequence of measurable functions converging pointwise of f. Suppose $|f_n| \leqslant g$ and that $\lim_{n\to\infty} \int g\, d\mu_n = \int g\, d\mu < \infty$. Then

$$\lim_{n\to\infty} \int f_n\, d\mu_n = \int f\, d\mu$$

A measurable space (X, B) is a standard Borel space if there exists a complete metric space Y such that B is σ-isomorphic to the Borel σ-field B_Y of Y, i.e., there is a bijection from B to B_Y which preserves countable set operations. The measurable spaces (Ω, S) and $(X(\Lambda), B_\Lambda)$ considered in this paper are standard Borel spaces. The following proposition has been used by Parthasarathy (ref. 13, p. 145) and Preston (ref. 3, p. 27). We provide a short proof for the convenience of the reader.

Proposition 2.3. Let X be uncountable and (X, B) a standard Borel space. There exists a countable field $B_0 \subset B$ such that $B = \sigma(B_0)$ and such that, if $\mu\colon B_0 \to [0, 1]$ is a finitely additive probability measure on B_0, then μ has a unique extension to a (countably additive) probability measure on (X, B).

Proof. (X, B) is isomorphic as a measure space to $\prod_{i=1}^{\infty} \{0, 1\}$ with the product Borel σ-field. Let B_n be the finite σ-field generated by the first n factors. Then $\bigcup_{n=1}^{\infty} B_n$ is a countable field. Any finitely additive probability measure on $\bigcup_{n=1}^{\infty} B_n$ is consistent on $\{B_n\}$. The result now follows by the Kolmogorov extension theorem. ∎

3. PRINCIPAL RESULTS

Lemma 3.1. There exist functions g_1, g_2, g_3 on U_∞, integrable with respect to any tempered Gibbs state, such that for all k sufficiently large, the following hold:

(a) $(1/|\Lambda_k|) \int |x \cap \Lambda_k| \, \mu_{\Lambda_k}(dx \mid s) \leqslant g_1(s)$.

(b) $(1/|\Lambda_k|) \left| \int W_{\Lambda_k}(x \mid s) \, \mu_{\Lambda_k}(dx \mid s) \right| \leqslant g_2(s)$.

(c) $(1/|\Lambda_k|) \left| \int H_{\Lambda_k}(x \mid s) \, \mu_{\Lambda_k}(dx \mid s) \right| \leqslant g_3(s)$.

Remark 3.1. The integrable bounds in Lemma 3.1 may be chosen to hold for all k; we find bounds only for all large values of k in order to streamline the proof.

Proof. Observe that for any function f on $X(\Lambda_k)$,

$$\int f(x) \, \mu_{\Lambda_k}(dx \mid s) = \frac{\int f(x) \exp[-\beta W_{\Lambda_k}(x \mid s)] \, \mu_{\Lambda_k}(dx)}{\int \exp[-\beta W_{\Lambda_k}(x \mid s)] \, \mu_{\Lambda_k}(dx)} \tag{3.1}$$

Let $\varepsilon > 0$ and $s \in U_n$. In what follows we identify Λ_k, Λ_m, and $\partial\Lambda \equiv \partial\Lambda_k$, as in Lemma 2.1. Let

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \subset \Lambda_m \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

Then, using the product structure of ν_{Λ_k},

$$\begin{aligned} &\int \exp[-\beta W_{\Lambda_k}(x \mid s)] \, \mu_{\Lambda_k}(dx) \\ &\quad \geqslant \frac{1}{Z_{\Lambda_k}(\varnothing)} \int_{X(\Lambda_k)} \chi(x) \exp[-\beta W_{\Lambda_k}(x \mid s)] \exp[-\beta H_{\Lambda_k}(x)] \, \nu_{\Lambda_k}(dx) \\ &\quad \geqslant \frac{1}{Z_{\Lambda_k}(\varnothing)} \int_{\{\varnothing\}} \int_{X(\Lambda_m)} \exp[-\beta W_{\Lambda_k}(x \mid s \cap \Lambda_k^c)] \\ &\quad\quad \times \exp[-\beta H_{\Lambda_m}(x)] \, \nu_{\Lambda_m}(dx) \, \nu_{\Lambda_k \setminus \Lambda_m}(dy) \\ &\quad \geqslant \int \exp(-\beta \varepsilon n \, |x_{\Lambda_m}|) \, \mu_{\Lambda_m}(dx) \frac{Z_{\Lambda_m}(\varnothing)}{Z_{\Lambda_k}(\varnothing)} \end{aligned}$$

Therefore, by Jensen's inequality,

$$\begin{aligned} &\ln \int \exp[-\beta W_{\Lambda_k}(x \mid s)] \, \mu_{\Lambda_k}(dx) \\ &\quad\quad \geqslant -\beta \varepsilon n \int |x_{\Lambda_m}| \, \mu_{\Lambda_m}(dx) + \ln Z_{\Lambda_m}(\varnothing) - \ln Z_{\Lambda_k}(\varnothing) \end{aligned} \tag{3.3}$$

We next bound $\int |x_{\Lambda_m}| \, \mu_{\Lambda_m}(dx)$ using Ruelle's probability estimates (Proposition 2.1):

$$\frac{1}{|\Lambda_m|}\int |x_{\Lambda_m}|\,\mu_{\Lambda_m}(dx)$$
$$=\int_0^{\infty}\mu_{\Lambda_m}\{x_{\Lambda_m}: |x_{\Lambda_m}|>y\,|\Lambda_m|\}\,dy$$
$$\leqslant\int_0^{(\delta/\gamma)^{1/2}}1\,dy+\int_{(\delta/\gamma)^{1/2}}^{\infty}\exp\{-(\gamma y^2-\delta)\,|\Lambda_m|\}\,dy\leqslant\left(\frac{\delta}{\gamma}\right)^{1/2}+\left(\frac{\pi}{4\gamma\,|\Lambda_m|}\right)^{1/2} \tag{3.4}$$

where δ and γ are the constants appearing in Proposition 2.1. Combining (3.3) and (3.4) gives

$$\ln\int\exp[-\beta W_{\Lambda_k}(x\,|\,s)]\,\mu_{\Lambda_k}(dx)$$
$$\geqslant -\beta\varepsilon n\left[\left(\frac{\delta}{\gamma}\right)^{1/2}|\Lambda_k|+\left(\frac{\pi\,|\Lambda_k|}{4\gamma}\right)^{1/2}\right]+\ln Z_{\Lambda_m}(\varnothing)-\ln Z_{\Lambda_k}(\varnothing) \tag{3.5}$$

To bound the numerator in (3.1), observe that for any $c>0$, and any union Λ of unit cubes in Λ_k,

$$\int\exp(c\,|x_\Lambda|)\,\mu_{\Lambda_k}(dx)$$
$$=\int_0^{\infty}\mu_{\Lambda_k}\{x_{\Lambda_k}: \exp(c\,|x_\Lambda|)>y\}\,dy$$
$$<\int_0^{\exp[(2c^2|\Lambda|)/\gamma]}1\,dy+\int_{\exp[(2c^2|\Lambda|)/\gamma]}^{\infty}\exp\left\{-\gamma\frac{(\ln y)^2}{c^2\,|\Lambda|}+\delta\,|\Lambda|\right\}dy$$
$$<\exp\left(\frac{2c^2\,|\Lambda|}{\gamma}\right)+[\exp(\delta\,|\Lambda|)]\int_{\exp[(2c^2|\Lambda|)/\gamma]}^{\infty}y^{-2}\,dy$$
$$<2\exp\left(\frac{\delta+2c^2}{\gamma}\,|\Lambda|\right) \tag{3.6}$$

For any $a\geqslant 0$, it follows from (3.6) and Lemma 2.1 that

$$\int\exp(a\,|x|)\exp[-\beta W_{\Lambda_k}(x\,|\,s)]\,\mu_{\Lambda_k}(dx)$$
$$\leqslant\int\exp[\beta D_k(s)\,|x_{\partial\Lambda}|]\exp[(\beta n\varepsilon+a)\,|x_{\Lambda_k}|]\,\mu_{\Lambda_k}(dx)$$
$$\leqslant\left(\int\exp[2\beta D_k(s)\,|x_{\partial\Lambda}|]\,\mu_{\Lambda_k}(dx)\right)^{1/2}\left(\int\exp[2(\beta n\varepsilon+a)\,|x|]\,\mu_{\Lambda_k}(dx)\right)^{1/2}$$
$$\leqslant 2\exp\left[\left(\frac{4\beta^2 D_k(s)^2}{\gamma}+\frac{\delta}{2}\right)|\partial\Lambda|+\left(\frac{4(\beta n\varepsilon+a)^2}{\gamma}+\frac{\delta}{2}\right)|\Lambda_k|\right] \tag{3.7}$$

Using Jensen's inequality and (3.1) gives

$$\begin{aligned}
&\int |x_{\Lambda_k}|\, \mu_{\Lambda_k}(dx \mid s)\\
&\leqslant \ln \int \exp(|x_{\Lambda_k}|)\, \mu_{\Lambda_k}(dx \mid s)\\
&\leqslant \ln \int \exp |x| \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx) - \ln \int \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx)
\end{aligned} \tag{3.8}$$

Combining (3.8) with (3.5) and (3.7) with $a=1$ gives

$$\begin{aligned}
&\frac{1}{|\Lambda_k|} \int |x_{\Lambda_k}|\, \mu_{\Lambda_k}(dx \mid s)\\
&\quad \leqslant \left(\frac{4\beta^2 D_k(s)^2}{\gamma} + \frac{\delta}{2}\right) \frac{|\partial \Lambda|}{|\Lambda_k|} + \frac{4(\beta n \varepsilon + 1)^2}{\gamma} + \frac{\delta}{2} + \frac{\ln 2}{|\Lambda_k|}\\
&\quad\quad + \beta \varepsilon n \left[\left(\frac{\delta}{\gamma}\right)^{1/2} + \left(\frac{\pi}{4\gamma |\Lambda_k|}\right)^{1/2}\right] - \frac{|\Lambda_m|}{|\Lambda_k|} \frac{1}{|\Lambda_m|} \ln Z_{\Lambda_m}(\varnothing)\\
&\quad\quad + \frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(\varnothing)
\end{aligned} \tag{3.9}$$

By Lemma 2.1, $D_k(s) \leqslant Cn(\ln k)^{1/2}$. Therefore the right side of (3.9) is a quadratic polynomial in n:

$$C_2(k) n^2 + C_1(k) n + C_0(k)$$

where $0 \leqslant C_i \equiv \sup_k C_i(k) < \infty$ for $i = 0, 1, 2$ and

$$n \equiv n(s) \equiv \min\{m \in \mathbf{Z}\colon s \in U_m\} \tag{3.10}$$

Define with (3.10)

$$g_1(s) = C_2 n^2 + C_1 n + C_0$$

If μ is a tempered Gibbs state, it is easy to show, using Proposition 2.1, that there exists a constant D such that

$$\mu(U_m^c) \leqslant D \exp(-\gamma m^2) \tag{3.11}$$

for all m sufficiently large. Thus

$$\int g_1(s)\, \mu(ds) \leqslant \sum_{i=0}^{2} C_i \sum_{m=1}^{\infty} m^i \mu(U_{m-1}^c) < \infty \tag{3.12}$$

To prove part b, observe that, by Lemma 2.1,

$$\frac{1}{|\Lambda_k|}\int W_{\Lambda_k}(x\,|\,s)\,\mu_{\Lambda_k}(dx\,|\,s)$$

$$\geqslant -\frac{1}{|\Lambda_k|}\int D_k(s)\,|x_{\partial\Lambda}|\,\mu_{\Lambda_k}(dx\,|\,s)-\int \varepsilon\,\frac{|x_{\Lambda_k}|}{|\Lambda_k|}\,\mu_{\Lambda_k}(dx\,|\,s) \qquad (3.13)$$

From part a, the second integral on the right is bounded below by $-\varepsilon g_1(s)$. To bound the first integral on the right side of (3.13), notice that by Jensen's inequality and (3.1)

$$\int D_k(s)\,|x_{\partial\Lambda}|\,\mu_{\Lambda_k}(dx\,|\,s)$$

$$\leqslant \ln\int \exp[D_k(s)\,|x_{\partial\Lambda}|]\,\exp[-\beta W_{\Lambda_k}(x\,|\,s)]\,\mu_{\Lambda_k}(dx)$$

$$-\ln\int \exp[-\beta W_{\Lambda_k}(x\,|\,s)]\,\mu_{\Lambda_k}(dx) \qquad (3.14)$$

Applying (3.5), (3.6), and Lemma 2.1 as before shows that the right side of (3.14) is bounded by a polynomial in $n(s)$ which is integrable with respect to any tempered Gibbs state.

On the other hand, by Jensen's inequality and (3.1),

$$\int \beta W_{\Lambda_k}(x\,|\,s)\,\mu_{\Lambda_k}(dx\,|\,s)$$

$$\leqslant \ln\int \exp[\beta W_{\Lambda_k}(x\,|\,s)]\,\mu_{\Lambda_k}(dx\,|\,s)$$

$$=\ln\frac{\int \exp[+\beta W_{\Lambda_k}(x\,|\,s)]\,\exp[-\beta W_{\Lambda_k}(x\,|\,s))]\,\mu_{\Lambda_k}(dx)}{\int \exp[-\beta W_{\Lambda_k}(x\,|\,s))]\,\mu_{\Lambda_k}(dx)}$$

$$\leqslant \beta\varepsilon n\left[\left(\frac{\delta}{\gamma}\right)^{1/2}|\Lambda_k|+\left(\frac{\pi\,|\Lambda_k|}{4\gamma}\right)^{1/2}\right]-\ln Z_{\Lambda_m}(\varnothing)+\ln Z_{\Lambda_k}(\varnothing) \qquad (3.15)$$

where the last inequality comes from (3.5). Dividing both sides of (3.15) by $\beta|\Lambda_k|$ shows that $(1/|\Lambda_k|)\int W_{\Lambda_k}(x\,|\,s)\,\mu_{\Lambda_k}(dx\,|\,s)$ is bounded above by a linear function in $n(s)$ with coefficients bounded in k. Hence it is bounded by a function $g_2(s)$ integrable with respect to any tempered Gibbs state.

By the stability of $H(x)$,

$$\frac{1}{|\Lambda_k|}\int H_{\Lambda_k}(x\,|\,s)\,\mu_{\Lambda_k}(dx\,|\,s)\geqslant\frac{1}{|\Lambda_k|}\int -B\,|x|+W_{\Lambda_k}(x\,|\,s)\,\mu_{\Lambda_k}(dx\,|\,s) \qquad (3.16)$$

The integral on the right is bounded below by a linear combination of the functions $g_1(s)$ and $g_2(s)$ from parts a and b.

To find an upper bound, write

$$\begin{aligned}
&\int \beta H_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s) \\
&\qquad \leqslant \ln \int \exp[\beta H_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx \mid s) \\
&\qquad = \ln \frac{\int \exp[+\beta H_{\Lambda_k}(x \mid s)] \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx)}{\int \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx)} \\
&\qquad = \ln \frac{\int \exp[+\beta H_{\Lambda_k}(x)] \exp[-\beta H_{\Lambda_k}(x)]\, \nu_{\Lambda_k}(dx)}{Z_{\Lambda_k}(\varnothing) \int \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx)} \\
&\qquad \leqslant \beta \varepsilon n \left[\left(\frac{\delta}{\gamma}\right)^{1/2} |\Lambda_k| + \left(\frac{\pi |\Lambda_k|}{4\gamma}\right)^{1/2} \right] - \ln Z_{\Lambda_m}(\varnothing) + |\Lambda_k| \qquad (3.17)
\end{aligned}$$

where in the last inequality we have used (3.5). Dividing both sides (3.17) by $\beta |\Lambda_k|$ shows that

$$\frac{1}{|\Lambda_k|} \left| \int H_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s) \right| \tag{3.18}$$

is bounded by a linear function of $n(s)$ with coefficients bounded in k and (3.18) is therefore bounded by an integrable function of s. ∎

Lemma 3.2. (a) For any $s \in U_\infty$

$$\lim_{k \to \infty} \int \frac{W_{\Lambda_k}(x \mid s)}{|\Lambda_k|}\, \mu_{\Lambda_k}(dx \mid s) = 0$$

(b) For any tempered Gibbs state μ

$$\lim_{k \to \infty} \int \frac{W(x_{\Lambda_k} \mid x_{\Lambda_k^c})}{|\Lambda_k|}\, \mu(dx) = 0$$

Proof. Part b follows from part a, Lemma 3.1b, and the dominated convergence theorem. From (3.15)

$$\limsup_{k \to \infty} \int \frac{W_{\Lambda_k}(x \mid s)}{|\Lambda_k|}\, \mu_{\Lambda_k}(dx \mid s) \leqslant \varepsilon n(s)(\delta/\gamma)^{1/2} \tag{3.19}$$

where we have used the same notation as in the proof of Lemma 3.1. Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{k\to\infty} \int \frac{W_{\Lambda_k}(x \mid s)}{|\Lambda_k|}\, \mu_{\Lambda_k}(dx \mid s) \leqslant 0 \tag{3.20}$$

From (3.5), (3.13), (3.14), and Lemma 3.1a

$$\begin{aligned} &\int W_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s) \\ &\quad \geqslant -\varepsilon g_1(s) |\Lambda_k| - \ln \int \exp[D_k(s)) |x_{\partial\Lambda}|] \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx) \\ &\qquad -\beta\varepsilon n \left[\left(\frac{\delta}{\gamma}\right)^{1/2} |\Lambda_k| + \left(\frac{\pi |\Lambda_k|}{4\gamma}\right)^{1/2} \right] + \ln Z_{\Lambda_m}(\varnothing) - \ln Z_{\Lambda_k}(\varnothing) \end{aligned} \tag{3.21}$$

It is necessary to bound the integral on the right side of (3.21) differently than in the proof of Lemma 3.1. We have

$$\begin{aligned} &\int \exp[D_k(s) |x_{\partial\Lambda}|] \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx) \\ &\quad \leqslant \int \exp[(\beta+1) D_k(s) |x_{\partial\Lambda}|] \exp(\beta n \varepsilon |x_{\Lambda_k}|)\, \mu_{\Lambda_k}(dx) \\ &\quad = \int \exp[(\beta+1) D_k(s) |x_{\partial\Lambda}|]\, \tilde{\mu}_{\Lambda_k}(dx) \frac{Z_{\Lambda_k}(\beta, h+\beta\varepsilon n, \varnothing)}{Z_{\Lambda_k}(\beta, h, \varnothing)} \end{aligned} \tag{3.22}$$

where $\tilde{\mu}_{\Lambda_k}$ is the finite-volume Gibbs state for $s = \varnothing$ with h replaced by $h + \beta\varepsilon n$. By (3.6)

$$\begin{aligned} &\int \exp[D_k(s) |x_{\partial\Lambda}|] \exp[-\beta W_{\Lambda_k}(x \mid s)]\, \mu_{\Lambda_k}(dx) \\ &\quad \leqslant 2 \exp\left\{ \left(\frac{2(\beta+1)^2 D_k(s)^2}{\tilde{\gamma}} + \tilde{\delta} \right) |\partial\Lambda| \right\} \frac{Z_{\Lambda_k}(\beta, h+\beta\varepsilon n, \varnothing)}{Z_{\Lambda_k}(\beta, h, \varnothing)} \end{aligned} \tag{3.23}$$

where $\tilde{\delta}$ and $\tilde{\gamma}$ are the constants from Proposition 2.1 for h replaced by $h + \beta\varepsilon n$. Combining (3.23) and (3.21) gives

$$\begin{aligned} &\int \frac{W(x_{\Lambda_k} \mid s)}{|\Lambda_k|}\, \mu_{\Lambda_k}(dx \mid s) \\ &\quad \geqslant -\beta\varepsilon n \left[\left(\frac{\delta}{\gamma}\right)^{1/2} + \left(\frac{\pi}{2\gamma |\Lambda_k|}\right)^{1/2} \right] + \frac{|\Lambda_m|}{|\Lambda_k|} \frac{1}{|\Lambda_m|} \ln Z_{\Lambda_m}(\varnothing) - \varepsilon g_1(s) \\ &\qquad - \left(\frac{2(\beta+1)^2 D_k(s)^2}{\tilde{\gamma}} + \tilde{\delta} \right) \frac{|\partial\Lambda|}{|\Lambda_k|} - \frac{\ln 2}{|\Lambda_k|} - \frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(\beta, h+\beta\varepsilon n, \varnothing) \end{aligned} \tag{3.24}$$

Therefore,

$$\liminf_{k\to\infty} \int \frac{W(x_{\Lambda_k} \mid s)}{|\Lambda_k|} \mu_{\Lambda_k}(dx \mid s) \geqslant -\varepsilon\beta n(s) \left(\frac{\delta}{\gamma}\right)^{1/2} - \varepsilon g_1(s)$$
$$+ P(\beta, h) - P(\beta, h + \beta\varepsilon n) \qquad (3.25)$$

By continuity of the pressure[10] in h and since $\varepsilon > 0$ is arbitrary,

$$\liminf_{k\to\infty} \int \frac{W(x_{\Lambda_k} \mid s)}{|\Lambda_k|} \mu_{\Lambda_k}(dx \mid s) \geqslant 0 \qquad (3.26)$$

Inequalities (3.20) and (3.26) establish part b. ▮

Corollary 3.1. For any $s \in U_\infty$,

$$\lim_{k\to\infty} \frac{\ln Z_{\Lambda_k}(s)}{|\Lambda_k|} = P(\beta, h)$$

Proof. For any k,

$$Z_{\Lambda_k}(\varnothing) = \int_{X(\Lambda_k)} \{\exp[\beta W_{\Lambda_k}(x \mid s)]\} \frac{\exp[-\beta H_{\Lambda_k}(x \mid s)]}{Z_{\Lambda_k}(s)} \nu_{\Lambda_k}(dx) \cdot Z_{\Lambda_k}(s) \qquad (3.27)$$

Taking logarithms and using Jensen's inequality gives

$$\ln Z_{\Lambda_k}(\varnothing) \geqslant \ln Z_{\Lambda_k}(s) + \beta \int W_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s) \qquad (3.28)$$

From Lemma 3.2a,

$$\limsup_{k\to\infty} \frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(s) \leqslant P(\beta, h) \qquad (3.29)$$

Assuming k is sufficiently large and using the same notation as in the proof of Lemma 3.1,

$$Z_{\Lambda_k}(s) \geqslant \int_{X(\Lambda_k)} \{\exp[-\beta H_{\Lambda_k}(x \mid s)]\}\, \chi(x)\, \nu_{\Lambda_k}(dx)$$
$$\geqslant \int \exp[-\beta\varepsilon n(s) |x_{\Lambda_m}|]\, \mu_{\Lambda_m}(dx) \cdot Z_{\Lambda_m}(\varnothing) \qquad (3.30)$$

Thus, using Jensen's inequality again shows that

$$\ln Z_{\Lambda_k}(s) \geqslant \ln Z_{\Lambda_m}(\varnothing) - \varepsilon n\beta \int |x_{\Lambda_m}| \, \mu_{\Lambda_m}(dx)$$

Applying Lemma 3.1a gives

$$\frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(s) \geqslant \frac{|\Lambda_m|}{|\Lambda_k|} \frac{1}{|\Lambda_m|} \ln Z_{\Lambda_m}(\varnothing) - \varepsilon n\beta \frac{|\Lambda_m|}{|\Lambda_k|} g_1(\varnothing) \tag{3.31}$$

Thus

$$\liminf_{k\to\infty} \frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(s) \geqslant P(\beta, h) - \varepsilon n\beta g_1(\varnothing)$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{k\to\infty} \frac{1}{|\Lambda_k|} \ln Z_{\Lambda_k}(s) \geqslant P(\beta, h) \tag{3.32}$$

Combining (3.32) and (3.29) proves the corollary. ▮

We state Lemma 3.3 below without proof.

Lemma 3.3. Let Λ be a bounded Borel set, $F \in \tilde{\mathcal{B}}_\Lambda$, $n \geqslant 1$, and let I_1 and I_2 be closed intervals on the real line with I_1 to the right of zero. Then:

(a) $\pi_\Lambda(F \mid s \cap \Lambda_k)(\beta, h) \to \pi_\Lambda(F \mid s)(\beta, h)$ uniformly for all $s \in U_n$, $\beta \in I_1$, and $h \in I_2$ as $k \to \infty$.

(b) If $\Lambda \equiv \Lambda_L$ for some integer L, $\pi_\Lambda(H_\Lambda(x) \mid s \cap \Lambda_k)(\beta, h) \to \pi_\Lambda(H_\Lambda(x) \mid s)(\beta, h)$ uniformly for all $s \in U_n$, $\beta \in I_1$, as $k \to \infty$.

Theorem 3.1. Let $Q \equiv Q_0 = \Lambda_{k=1}$ be the unit cube centered at the origin.

(a) The expectation

$$\int H_Q(x) + \tfrac{1}{2} W_Q(x \mid x_{Q^c}) \, \mu(dx) \tag{3.33}$$

is the same for any translation-invariant, tempered Gibbs state μ for H, β_0, h, if and only if $P(\beta, h)$ is continuously differentiable at β_0.

(b) The expectation

$$\int |x \cap Q| \, \mu(dx)$$

is the same for any translation-invariant, tempered Gibbs state μ for H, β, h_0, if and only if $P(\beta, h)$ is continuously differentiable at h_0.

Remark 3.2. Theorem 3.1 may be modified. In Eq. (2.1), one may assume, if desired, that $h = \hat{h}/\beta$ for some "chemical potential" $\hat{h}$ independent of β. In this way βh is independent of β. With this convention, β is the coefficient of the particle interaction energy and h is, independently, the coefficient of the particle number in $P(\beta, h)$. Note also that

$$\int H_Q(x) + \frac{1}{2} W_Q(x \mid x_{Q^c})\, \mu(dx) = \lim_{k \to \infty} \frac{1}{|\Lambda_k|} \int H_{\Lambda_k}(x)\, \mu(dx)$$

by translation invariance of μ and Lemma 3.2, so that part a of Theorem 3.1 may be reformulated. The restriction that $Q = \Lambda_1$, the unit cube, in Theorem 3.1 may be relaxed. Q can be chosen to be any geometric solid whose translates partition $\mathbf{R}^d$, such as a rectangular solid. The underlying lattice $\mathbf{Z}^d$ must then be replaced with another lattice; Λ_k then becomes a union of translates of Q for each k, U_∞ is then changed, etc.

Proof. (a) Since $P(\beta, h)$ is a convex function of β, P is differentiable on a dense subset of the positive real line. Suppose that P is differentiable at β. For any k, $(1/|\Lambda_k|) \ln Z_{\Lambda_k}(s)$ is convex and differentiable with respect to β for any $s \in U_\infty$. From Corollary 3.1, it follows that for any point β where P is differentiable,

$$\frac{dP}{d\beta} = \lim_{k \to \infty} \frac{1}{|\Lambda_k|} \int H_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s)$$

Let μ be a translation-invariant, tempered Gibbs state. From the Lebesgue dominated convergence theorem and Lemma 3.1 we have

$$\frac{dP}{d\beta} = \lim_{k \to \infty} \frac{1}{|\Lambda_k|} \iint H_{\Lambda_k}(x \mid s)\, \mu_{\Lambda_k}(dx \mid s)\, \mu(ds)$$

By the definition of a Gibbs state and Lemma 3.2,

$$\frac{dP}{d\beta} = \lim_{k \to \infty} \frac{1}{|\Lambda_k|} \int H_{\Lambda_k}(x \mid x_{\Lambda_k^c})\, \mu(dx) = \lim_{k \to \infty} \frac{1}{|\Lambda_k|} \int H_{\Lambda_k}(x)\, \mu(dx) \tag{3.34}$$

Now write

$$\begin{aligned} H_{\Lambda_k}(x) &= \sum_i \left[H_{Q_i}(x) + \tfrac{1}{2} W(x_{Q_i} \mid x_{Q_i^c \cap \Lambda_k})\right] \\ &= \sum_i \left[H_{Q_i}(x) + \tfrac{1}{2} W(x_{Q_i} \mid x_{Q_i^c})\right] - \tfrac{1}{2} W(x_{\Lambda_k} \mid x_{\Lambda_k^c}) \end{aligned} \tag{3.35}$$

where the sums are over all i such that $Q_i \subset \Lambda_k$. Combining (3.34) and (3.35) and using the translation invariance of μ gives

$$\frac{dP}{d\beta} = \int H_Q(x) + \frac{1}{2} W(x_Q \mid x_{Q^c})\, \mu(dx) - \frac{1}{2} \lim_{k \to \infty} \int \frac{W(x_{\Lambda_k} \mid x_{\Lambda_k^c})}{|\Lambda_k|}\, \mu(dx) \tag{3.36}$$

From Lemma 3.2

$$\frac{dP}{d\beta} = \int H_Q(x) + \frac{1}{2} W(x_Q \mid x_{Q^c})\, \mu(dx) \tag{3.37}$$

Thus (3.33) is the same for all translation-invariant Gibbs states if P is differentiable at β_0.

Let

$$g(x) = H_Q(x) + \tfrac{1}{2} W(x_Q \mid x_{Q^c}) \tag{3.38}$$

and let $\{\beta_m\}$ be chosen so that $\beta_m \downarrow \beta_0$ and such that $P(\cdot, h)$ is differentiable at each β_m. Let $d^rP/d\beta$ and $d^lP/d\beta$ denote, respectively, right- and left-hand derivatives of P. Then

$$\frac{d^rP}{d\beta}(\beta_0, h) \leqslant \lim_{m \to \infty} \frac{dP}{d\beta}(\beta_m, h) = \lim_{m \to \infty} \int g(x)\, \mu_m(dx) \tag{3.39}$$

by (3.37), where μ_m is a translation-invariant, tempered Gibbs state for H, h, β_m.

The next step is to show that for some subsequence of $\{\mu_m\}$, which we again denote by $\{\mu_m\}$,

$$\lim_{m \to \infty} \int g(x)\, \mu_m(dx) = \int g(x)\, \mu(dx) \tag{3.40}$$

where μ is a translation-invariant, tempered Gibbs state for H, β_0, h. Then by (3.39) and (3.40),

$$\frac{d^rP}{d\beta}(\beta_0, h) \leqslant \int g(x)\, \mu(dx) \tag{3.41}$$

An analogous inequality bounding $(d^lP/d\beta)(\beta_0, h)$ below, together with the assumption that $g(x)$ has the same expectation with respect to any translation-invariant Gibbs state at β_0, will prove that P is continuously differentiable at β_0.

Let $\tilde{A}_\Lambda$ be the countable field given by Proposition 2.3 for the σ-field $\tilde{B}_\Lambda$. Define

$$\tilde{A}_\infty = \bigcup_k \tilde{A}_{\Lambda_k} \tag{3.42}$$

Since $\tilde{A}_\infty$ is countable, some subsequence of $\{\mu_m\}$, which we again denote by $\{\mu_m\}$, converges for each element of $\tilde{A}_\infty$. Define $\mu(A)$ by

$$\mu(A) = \lim_{m \to \infty} \mu_m(A) \tag{3.43}$$

By Proposition 2.3, for any fixed k, μ has a unique extension to $\tilde{B}_{A_k}$, which we again denote by μ. Let $F \in \tilde{B}_{A_k}$ and $s \in U_\infty$. Recall that $F' = \{x \in X(A): x \vee s \in F\}$ and in this case F' is independent of s. Then

$$\begin{aligned}
\pi_{A_k}(F \mid s) &\leqslant \int_{F'} \exp\{-\beta H_{A_k}(x \mid s)\}\, \nu_{A_k}(dx) \\
&\leqslant \int_{F'} \exp\left\{-\beta \frac{A}{|A_k|}\, |x|^2 + \beta[B + n(s)\, C(\ln_+ k)^{1/2}]\, |x|\right\} \nu_{A_k}(dx) \\
&\leqslant \max\left\{\exp\left\{-\beta \frac{A}{|A_k|}\, |x|^2 \right.\right. \\
&\qquad \left.\left. + \beta[B + n(s) C(\ln_+ k)^{1/2}]\, |x|\right\} : |x| \in R\right\} \nu_{A_k}(F') \\
&\equiv M(\beta, h, k, n(s))\, \nu_{A_k}(F')
\end{aligned} \tag{3.44}$$

where we have used Remark 2.2, superstability, the observation that $Z_A(s) \geqslant 1$ for all s and A, and

$$A \sum_{i \in A_k} |x_i|^2 \geqslant A \left(\sum_{i \in A_k} |x_i| \right)^2 |A_k|^{-1}$$

It follows from (3.44) that $\{\pi_{A_k}(F \mid s)(\beta_m, h)$: $m = 1, 2, 3,\ldots$, and $s \in U_n\}$, where $\beta_m \downarrow \beta_0$ as above, is uniformly absolutely continuous with respect to the measure on $\tilde{B}_{A_k}$ given by $\omega_k(F) \equiv \nu_{A_k}(F')$.

From Proposition 2.1 all tempered Gibbs states for a given value of β and h satisfy Ruelle's estimates for the same values of γ and δ. It follows from the proofs in ref. 9 that the same values of γ and δ may be selected for the entire sequence of tempered Gibbs states $\{\mu_m\}$ given in (3.39) corresponding to $\beta_m \downarrow \beta_0$ (in fact, $\gamma = \beta_0 A/4$ may be used).

Let $\varepsilon > 0$ be given. Choose n so that $\mu_m(U_n^c) < \varepsilon/2$ for all m. Choose $\eta > 0$ so that $\pi_{A_k}(F \mid s)(\beta_m, h) < \varepsilon/2$ whenever $\omega_k(F) < \eta$ and $s \in U_n$. Then

$$\begin{aligned}
\mu_m(F) &= \mu_m(\pi_{A_k}(F \mid s)(\beta_m, h)) \\
&= \int_{U_n} \pi_{A_k}(F \mid s)(\beta_m, h)\, \mu_m(ds) + \int_{U_n^c} \pi_{A_k}(F \mid s)(\beta_m, h)\, \mu_m(ds) < \varepsilon
\end{aligned} \tag{3.45}$$

Thus, given any k, the measures $\{\mu_m\}$ restricted to $\tilde{B}_{A_k}$ are uniformly absolutely continuous with respect to ω_k.

Let $A \subset \mathbf{R}^d$ be a bounded Borel set and let $F \in \tilde{B}_A$. Without loss of generality, we may assume $A = A_k$ for some k. Let $\varepsilon > 0$ and choose η as in (3.45). Since $\sigma(\tilde{A}_A) = \tilde{B}_A$, there exists an $A \in \tilde{A}_A$ such that $\mu(A \triangle F) < \varepsilon$ and $\omega_k(A \triangle F) < \eta$. Here $A \triangle F = (A \backslash F) \cup (F \backslash A)$. By the triangle inequality,

$$\begin{aligned} |\mu_m(F) - \mu(F)| &\leqslant |\mu_m(A) - \mu(A)| + \mu_m(A \triangle F) + \mu(A \triangle F) \\ &\leqslant |\mu_m(A) - \mu(A)| + 2\varepsilon \end{aligned} \tag{3.46}$$

It follows that

$$\mu(F) = \lim_{m \to \infty} \mu_m(F) \tag{3.47}$$

for all $F \in \tilde{B}_A$. Equation (3.47) shows that $\mu(F)$ is consistently defined on the increasing sequence of σ-fields $\{\tilde{B}_{A_k}\}$. Since these σ-fields generate the σ-field S, μ has a unique extension to a probability measure on (Ω, S), which we again denote by μ. The translation invariance of μ follows from the translation invariance of μ_m and standard arguments in measure theory.

We next prove that μ is a Gibbs state for β_0, H, h. It is routine to verify that

$$\pi_A(F \mid s)(\beta_m, h) \to \pi_A(F \mid s)(\beta_0, h) \tag{3.48}$$

for each $s \in U_\infty$, each A, and each measurable set F. By the triangle inequality,

$$\begin{aligned} &|\mu_m(\pi_A(F \mid s)(\beta_m, h)) - \mu(\pi_A(F \mid s)(\beta_0, h))| \\ &\quad \leqslant |\mu_m[\pi_A(F \mid s \cap A_k)(\beta_m, h) - \mu(\pi_A(F \mid s \cap A_k)(\beta_0, h)]| \\ &\qquad + |\mu_m[\pi_A(F \mid s)(\beta_m, h) - \pi_A(F \mid s \cap A_k)(\beta_m, h)]| \\ &\qquad + |\mu[\pi_A(F \mid s)(\beta_0, h) - \pi_A(F \mid s \cap A_k)(\beta_0, h)]| \end{aligned} \tag{3.49}$$

It follows from Lemma 3.3 and arguments similar to those leading to (3.45) that by choosing k sufficiently large, the last two terms on the right side of (3.49) can made arbitrarily small, uniformly in m. By Proposition 2.2 and (3.48) the first term on the right side of (3.49) converges to zeros as $m \to \infty$ for any fixed k.

Thus,

$$\mu_m(\pi_A(F \mid s)(\beta_m, h)) \to \mu(\pi_A(F \mid s)(\beta_0, h))$$

Since we also have

$$\mu_m(\pi_\Lambda(F|s)(\beta_m, h)) = \mu_m(F) \to \mu(F)$$

for any cylinder set F, it follows that μ is a Gibbs state. It is easy to check that μ is tempered, using the fact that the same constants γ and δ may be used for each μ_m.

It remains to verify (3.40). Note that (3.40) does not follow from (3.47), because $g(x)$ is an unbounded function of x. A detailed argument using Lemma 3.2, Lemma 3.3b, Proposition 2.2, and the ideas in the proof of Lemma 3.1 proves (3.40) and completes the proof of part a, i.e.,

$$\mu_m(H_Q(x) + \tfrac{1}{2} W(x_Q | x_{Q^c})) \to \mu(H_Q(x) + \tfrac{1}{2} W(x_Q | x_{Q^c}))$$

(b) The proof of b follows as in part a with the cylinder function $|x_Q|$ playing the role of $g(x)$ and h playing the role of β. ∎

Remark 3.3. Theorem 3.1 may be extended to deal with Gibbs states invariant under groups which preserve the algebra of measurable cylinder sets, other than the translation group on $\mathbf{R}^d$. For example, let G be a group of Euclidean motions on $\mathbf{R}^d$ containing a subgroup of the translation group. Assuming that Gibbs states invariant under G exist for each β and h, the proof of Theorem 3.1 may be modified to show that the pressure is differentiable with respect to β (resp. h) if and only if all Gibbs states invariant under G yield the same expected specific energy (resp. density of particles).

The following corollary is now immediate.

Corollary 3.2. Suppose the Gibbs state for H, β_0, h_0 is unique. Then the pressure $p(\beta, h)$ is continuously differentiable with respect to β and with respect to h at (β_0, h_0).

Corollary 3.3 below follows from the proof of Theorem 3.1.

Corollary 3.3. Let μ_m be a translation-invariant, tempered Gibbs state for H, β_m, h and suppose $\beta_m \to \beta_0 > 0$. Then the sequence $\{\mu_m\}$ has a subsequence whose limit on any cylinder set F is $\mu(F)$, where μ is a translation-invariant, tempered Gibbs state for H, β_0, h. An analogous statement holds when $h_m \to h$, and β is fixed.

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